Phys 410
Fall 2014
Lecture #11 Summary
7 October, 2014

We began to talk about the calculus of variations. The calculus of variations is used to find extremum values of integral functionals. An example is a calculation of the shortest distance between two points in a plane. One can write the distance in terms of an integral over the path from the designated starting point (x_1, y_1) to the designated end point (x_2, y_2) as $L = \int_1^2 ds = \int_1^2 \sqrt{dx^2 + dy^2}$. If we (arbitrarily) treat the x coordinate as the independent variable we can write the integral as $L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$, where we have written $(dy/dx)^2$ as $(y')^2$. Our objective is to find the path $y(x)$ that minimizes this integral. This is a problem in the calculus of variations.

A second example is Fermat's principle. This is the problem of how light rays propagate from point 1 to point 2 through a variable dielectric medium characterized by an index of refraction that varies with position in a plane as $n(x, y)$. The light moves with variable speed $v = c/n(x, y)$. Fermat's principle says that light will take the path that minimizes the time to travel between the two points: $time(1 \to 2) = \frac{1}{c}\int_{x_1}^{x_2} n(x, y)\sqrt{1 + (y')^2}\, dx$. Again we need to find the path $y(x)$ that minimizes this integral. This is another problem in the calculus of variations.

The Euler-Lagrange equation is derived by assuming that there is an infinite family of "wrong" trajectories between points 1 and 2 parameterized by the function $\eta(x)$ and the constant α as $Y(x) = y(x) + \alpha\eta(x)$. The objective is to minimize the integral $S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx$, and this will be accomplished by taking $dS/d\alpha$ and setting it equal to zero. The result, after integrating by parts, is that the following expression must be satisfied for all points $x_1 \le x \le x_2$: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$, called the Euler-Lagrange equation.

Going back to the shortest-distance-in-a-plane problem, we see that the function f in this case is $f = \sqrt{1 + (y')^2}$. In this case f does not depend explicitly on$y$, hence we can write $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}} = C$, a constant. This can be reduced to $y'(x) = m$, where m is another constant. Integrating both sides with respect to x, we find $y(x) = mx + b$, which is the famous equation for a straight line. The Fermat's principle problem can be solved in a similar way once the index of refraction distribution $n(x, y)$, and the end points, are specified.

We then did the example of the Brachistochrone problem. A particle falls from rest under the influence of gravity following a frictionless track to a final location. The question is: what track design will get the particle to the final location in the shortest time? The particle starts at the origin (x=0, y=0) and falls to a point (x_2, y_2), with $x_2 > 0$ and $y_2 > 0$ (note that positive y is in the 'down' direction). The time to travel is given by $Time(1 \to 2) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{dx^2+dy^2}}{v}$. The speed is found from conservation of energy: $v = \sqrt{2gy}$, leading to $Time(1 \to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{1+(x\prime)^2}}{\sqrt{y}} dy$, where we are using the y-coordinate of the particle as the independent variable and $x' = dx/dy$. We are now looking for the trajectory $x(y)$ that minimizes the time $Time(1 \to 2)$. This integral will be made stationary when the integrand $f(x, x', y)$ obeys the Euler-Lagrange equation, which in this case is: $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$. The result is a differential equation for $x(y)$: $x' = \sqrt{\frac{y}{2a-y}}$, where a is a constant introduced from the Euler-Lagrange equation. We can integrate this equation with the change of variables $y = a(1 - \cos\theta)$, yielding $x = a(\theta - \sin\theta) + C$. This describes a cycloid curve (our cycloid is an upside-down version of the one on this web site). The particle making the shortest-time fall will follow the cycloid trajectory.

In general, it is not always possible to parameterize the trajectory of a particle with a simple one-to-one functional relationship such as $y(x)$ or $x(y)$. In this case one would like to parameterize the trajectory with functions such as $(x(u), y(u))$, where u acts as the parameter. The Euler-Lagrange equation can be generalized to handle this situation. Consider the integral $S = \int_{u_1}^{u_2} f[x(u), x'(u), y(u), y'(u), u]du$. To make it stationary will yield two Euler-Lagrange equations: $\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0$ and $\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y\prime} = 0$.

We then showed that Newton's second law of motion can be re-stated as a set of Euler-Lagrange equations for an integrand known as the Lagrangian $\mathcal{L} = T - U$, where T is the kinetic energy and U is the potential energy. The integral that is made stationary is called the action: $S = \int \mathcal{L}\, dt$. Hamilton's principle states that the actual motion of the particle will be the one that leaves this integral stationary. The Lagrangian can be written in terms of any set of unique (generalized) coordinates (q_1, q_2, q_3). One can define a generalized force as $\frac{\partial \mathcal{L}}{\partial q_i}$, and the generalized momentum as $\frac{\partial \mathcal{L}}{\partial q_i\prime}$. They are related through the Euler-Lagrange equation as "generalized force" = time rate of change of "generalized momentum". Note that these generalized quantities do not necessarily have the dimensions of force or momentum!

Feynman's path integral formulation of quantum mechanics considers all possible trajectories between the initial point and the final point. One calculates a transition amplitude as a sum over all trajectories of a weighting function. The weight of each trajectory is given the same magnitude, but a variable phase, as $e^{iS/\hbar}$, where S is the action for that trajectory and $\hbar$ is Planck's constant divided by 2π, which is sometimes known as the quantum of action. This is a generalization of Hamilton's principle, which of course specifies only a single classical trajectory.